Continuum percolation with discs having a distribution of radii

1984 J. Phys. A: Math. Gen. 17 L645

(http://iopscience.iop.org/0305-4470/17/12/004)

J. Phys. A: Math. Gen. 17 (1984) L645–L649. Printed in Great Britain

LETTER TO THE EDITOR

Continuum percolation with discs having a distribution of radii

Mohan K Phani† and Deepak Dhar‡

†Department of Physics, Indian Institute of Science, Bangalore 560012, India
‡Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Received 7 March 1984

Abstract. It is shown that for continuum percolation with overlapping discs having a distribution of radii, the net areal density of discs at percolation threshold depends non-trivially on the distribution, and is not bounded by any finite constant. Results of a Monte Carlo simulation supporting the argument are presented.

Although most of the physical systems modelled by percolation are basically continuum systems, the problem itself has been studied mostly in its discrete version (on lattices). Even in the continuum case, most of the interest has been centred on percolation of equal-sized discs (or spheres in three dimensions), mainly dealing with the question whether discrete and continuum percolation problems are in the same universality class (Vicsek and Kertesz 1981, Gawlinski and Stanley 1981). In physical systems, the requirement that all percolating units be of equal size is hardly satisfied. A special case in which percolation of basic units (say spheres) having a distribution of sizes becomes important is the phenomenon of phase inversion as observed in stirred mixtures of immiscible liquids (say oil and water). Consider oil being slowly added to water, and the mixture constantly agitated by a stirrer. If the total amount of oil added is small, it forms random-sized droplets in the background of water. On increasing the oil fraction, the system goes through an intermediate phase (in which both oil and water may be said to form spanning clusters) to a phase with water droplets dispersed in oil. The exact value of critical concentration of oil when the oil-in-water phase disappears depends on the distribution of radii of the oil droplets in water (this may be controlled by changing the stirring speed etc., see Clarke and Sawistowski 1978), the detailed dynamics of droplet coalescence etc. In this paper, we shall ignore the complications of dynamics and study the related geometrical problem of random percolation of overlapping discs with a distribution of radii.

Recent studies of continuum percolation of random-sized basic units have employed the Monte Carlo renormalisation group technique (Kertesz and Vicsek 1982) and the straightforward Monte Carlo method (Gawlinksi and Redner 1983). One of the results of Kertesz and Vicsek was that for a large class of distributions of disc radii, the average fractional area covered at the percolation threshold was 0.70 ± 0.02, and they conjectured that it might be 'constant for a large class of distributions'. For earlier references on this 'critical volume fraction' rule see Scher and Zallen (1970), and Pike and Seager (1974). In the following, we argue that his conjecture is only approximate.

0305-4470/84/120645 +05\$02.25

In fact, one can construct distributions for which the average fractional area covered by at least one of the discs is arbitrarily close to 1 at the percolation threshold. We present some Monte Carlo results in support of our arguments.

Consider continuum disc percolation in two dimensions (the argument is easily generalised to higher dimensions). The positions of centres of discs are uniformly distributed on a two-dimensional plane, $n(R)\,\mathrm{d}R\,\mathrm{d}A$ being the probability that the centre of the disc having radius between R and $R+\mathrm{d}R$ lies inside a small area $\mathrm{d}A$. The total area of discs per unit area of the plane is

$$\rho = \int \pi R^2 n(R)\,\mathrm{d}R, \tag{1}$$

ρ will be called the areal density of the discs. The average fractional area of the plane that is covered by at least one of the discs is easily shown to be $1-\exp(-\rho)$. In the case of percolation of discs of equal radii, it is easy to see that the areal density of the discs at the percolation threshold is independent of the size of the discs. Let this critical value of ρ be denoted by ρ^*. The numerical value of ρ^* is 1.20 ± 0.07, which corresponds to a critical covered area fraction $=0.70\pm0.02$ (Kertesz and Vicsek 1982). Constancy of the critical covered area fraction would imply that if $\rho=\rho^*$, the system would be at percolation threshold independent of the details of the distribution $n(R)$.

We consider, first, the special case when the radii of discs can take only two allowed values R_1 and R_2. Let ρ_1 be the areal density of discs of radius R_1. We assume that $\rho_1<\rho^*$, so that the system is subcritical and has a finite correlation length ξ_1. By length scaling, we can write $\xi_1=R_1 f(\rho_1)$, where $f(x)$ is the correlation length of a system of discs with radius 1 and density x. We now randomly drop discs of radius R_2 on the plane until the critical threshold is reached. Let the areal density of R_2-discs at this time be ρ_2. The critical areal density at percolation threshold for this two-valued radius distribution is

$$\rho_c=\rho_1+\rho_2. \tag{2}$$

The value of ρ_2, and hence ρ_c, depends on ρ_1 and R_2/R_1, and is in general expected to be a complicated function. Its value can be determined in some limiting cases.

If $R_2=R_1$, clearly ρ_c is equal to ρ^*, the critical density for single-sized disc percolation and $\rho_2=\rho^*-\rho_1$.

Consider now the case $R_2 \gg \xi_1 \gtrsim R_1$. We drop a single disc of radius R_2 on a plane with areal density ρ_1 of the R_1-disc. For large R_2, for most configurations, the cluster containing this R_2-disc and connected R_1 discs is roughly circular in shape with radius $(R_2+a\xi_1)$ (figure 1). Here a is a finite constant with numerical value $\simeq 1$. Thus the effect of the background of R_1-discs on the percolation of R_2-discs is to increase the effective radius of R_2-discs to a value $(R_2+a\xi_1)$. Two R_2-discs dropped near each other will usually belong to the same cluster if their centres are separated by a distance less than $2(R_2+a\xi_1)$. On the other hand, if the distance is much greater than this, the probability that there is a connecting path between them through R_1-disc clusters is very small.

Each R_2-disc gives rise to an approximately circular cluster of radius $(R_2+a\xi_1)$. Percolation occurs over macroscopic distances if these effective discs have sufficient overlap with each other to form an infinite cluster. It is thus easy to see that to a very good approximation the percolation threshold is reached when the areal density of

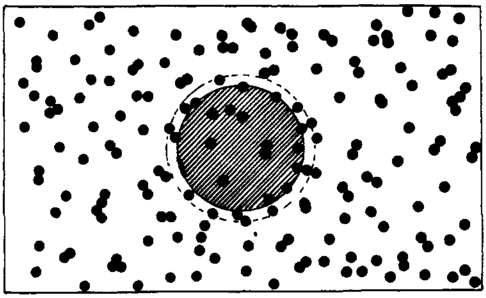

Figure 1. A disc of large radius dropped in the background of a finite areal density of small discs. The corresponding cluster is roughly circular in shape. The broken circle denotes the perimeter of the disc which approximates the cluster.

these effective discs is ρ^*. Hence

$$\rho_2 \simeq \rho^*(1 + a\xi_1/R_2)^{-2} \tag{3}$$

and

$$\rho_c \simeq \rho_1 + \rho^*(1 + a\xi_1/R_2)^{-2}. \tag{4}$$

As R_2 tends to infinity, ρ_c tends to the value $\rho_1 + \rho^*$. This result is easily seen to be correct independent of the approximations used to obtain equation (3). Since $0 < \rho_1 < \rho^*$, we get in this limit

$$\rho^* < \rho_c < 2\rho^*. \tag{5}$$

In the limit $R_s \ll R_1$, the same argument can be repeated after interchanging the roles of R_1 and R_2. Thus ρ_c tends to a value $(\rho_1 + \rho^*)$ as (R_2/R_1) tends to zero or infinity; but has a lower value $= \rho^*$ if $R_2/R_1 = 1$. This qualitative behaviour is shown in figure 2.

A similar argument with n types of discs with radii $R_i (i = 1$ to $n)$ satisfying $R_{i+1} \gg R_i f(\rho_i) \gg R_i$ shows that ρ_c can be arbitrarily close to $n\rho^*$, if the disc-sizes are widely different from each other. Thus the critical areal density ρ_c is not bounded by any finite constant for arbitrary distributions $n(R)$.

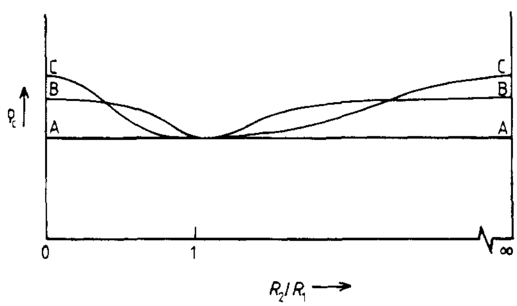

Figure 2. Schematic variation of the critical areal density ρ_c with the ratio of radii of discs (R_2/R_1) for several values of ρ_1. Curves A, B, C correspond to increasing values of $\rho_1: 0 = \rho_{1A} < \rho_{1B} < \rho_{1C} < \rho^*$, the critical density for single-sized disc percolation. For larger ρ_1, the value of ρ_c in the limits $R_2/R_1 = 0$ or ∞, is larger, but is reached more slowly.

We performed a Monte Carlo simulation to test the validity of the arguments presented above. We took a square of size $\sqrt{800}\times\sqrt{800}$ units (with periodic boundary conditions) and filled it randomly with discs of radius $R_1 = 1$ unit with average areal density ρ_1. We then determined ρ_2 by randomly placing discs of radius R_2 till a spanning cluster across the square is formed. To decrease the fluctuations in ρ_2, we average over several ($\geqslant 5$) realisations of R_2-discs keeping the configuration of R_1-discs fixed. To partially cancel the effect of fluctuations in the R_1-disc clusters, we compare ρ_2 with the values ρ_2', the average areal density of R_2-discs needed to reach the percolation threshold if $R_2 = R_1$. Finally we average over several (~ 10) configurations of R_1-discs.

The results of the simulation are summarised in table 1. We see that even for a fairly small value, 0.35, of R_2/R_1, ρ_2 is approximately equal to ρ_2' suggesting that $\rho_c \approx \rho^*$. This is in agreement with the observation of Kertesz and Vicsek. However, for an even smaller value $R_2/R_1 = 0.25$, we see that ρ_2 is significantly larger than ρ_2', in agreement with our conclusions presented above. If (R_2/R_1) is very small, ρ_2 should tend to ρ^*. We were unable to verify this conclusion, as the computer time becomes very large as the number of discs in the sample increases. (The total number of discs at criticality in our simulation is ~ 1500 for $R_2 = 0.35$ and approximately 2500 for $R_2 = 0.25$.) The value of $\rho^* = 1.13 \pm 0.01$ in our simulation is somewhat less than the value reported by Kertesz and Vicsek. This is due to finite size effect. Ideally, one would like to determine the critical threshold for several sizes of the lattice, and then extrapolate the results to infinite size limit. We have not carried out this procedure, mainly because of the large computer time involved. However, the finite size correction for $R_2/R_1 = 0.25$, should not differ too much from that for $R_2/R_1 = 0.35$, and clearly cannot account for the observed increase of ρ_2.

We thus conclude that the Monte Carlo experiments support our theoretical prediction that the critical areal density is significantly larger than ρ^*, if the radii of discs ave a large range of allowed values. However, this effect is significant only if the ratio of radii is typically bigger than three. In the case of continuous distribution of radii of the type studied by Kertesz and Vicsek, most of the weight of the distribution is concentrated between $\frac{1}{2}R_0$ and $2R_0$, where R_0 is the median radius of the distribution. This effect is thus too small to be observable in their Monte Carlo simulations.

Table 1. Results of the Monte Carlo simulation of continuum percolation in two dimensions with two sizes of discs. ρ_1 is the areal density of large-sized discs. ρ_2 is the average areal density of the smaller discs that have to be added to the sample in order to form a spanning cluster. ρ_2' is the average areal density that would have been needed if the new discs were of same radius as the earlier placed once ($R_2 = R_1$). The error bars are determined from the spread of observed values and do not take into account systematic corrections like the finite size effects.

Ratio of radii of discs	ρ_1	ρ_2	ρ_2'
0.35	0.55	0.604 ± 0.011	0.598 ± 0.006
	0.70	0.424 ± 0.013	0.437 ± 0.006
0.25	0.60	0.557 ± 0.016	0.508 ± 0.009
	0.70	0.514 ± 0.012	0.444 ± 0.009
	0.80	0.388 ± 0.023	0.301 ± 0.009

MKP would like to thank the theoretical physics group at Tata Institute of Fundamental Research for their hospitality during his stay.

References

Clarke S I and Sawistowski H 1978 *Trans. Inst. Chem. Eng.* **56** 50
Gawlinksi E T and Redner S 1983 *J. Phys. A: Math. Gen.* **16** 1063
Gawlinski E T and Stanley H E 1981 *J. Phys. A: Math. Gen.* **14** L291
Kertesz J and Vicsek T 1982 *Z. Phys.* B **45** 345
Pike G E and Seager C H 1974 *Phys. Rev.* B **10** 1621
Scher H and Zallen R 1970 *J. Chem. Phys.* **53** 3759
Vicsek T and Kertesz J 1981 *J. Phys. A: Math. Gen.* **10** L31